MATH 3963
NONLINEAR ODES WITH APPLICATIONS

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Contents

1. The index of a critical point

This topic explores some geometric and topological aspects of planar autonomous systems. We'll start with a 2×2 planar autonomous system, with a C^2 vector field (actually, this whole thing will go through with a C^0 vector field, but to avoid technical difficulties, we will just always assume that our vector field is as differentiable as it needs to be).

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} \tag{1.1}$$

Our next ingredient is a (special type of) plane curve. Let $\Gamma : [s_0, s_1] \to \mathbb{R}^2$ be a *simple, smooth, closed* plane curve oriented counterclockwise on which F has no critical points. By *simple* we mean that Γ has no self intersections, by *smooth*, we mean differentiable as many times as we want, and by *closed* we mean boundaryless. Boundaryless in the sense of a curve means that if $\Gamma(s_0) = (\gamma_1(s_0), \gamma_2(s_0))$ and $\Gamma(s_1) = (\gamma_1(s_1), \gamma_2(s_1))$, then $\Gamma(s_0) = \Gamma(s_1)$. We have that at each point of Γ, there is a (nonzero) vector $\begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$, and for each of these vectors, we can measure θ, the angle of inclination of $\begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$ with the horizontal, measured counter clockwise. Note that this means that

$$\tan(\theta) = \frac{g}{f}. \tag{1.2}$$

So in this sense we have that $\theta = \theta(s)$ a function of one variable, as we move through our curve Γ. Because the vector field is smooth (continuous) and because our curve is closed, we must have that $\theta(s_1) = \theta(s_0) + 2\pi n$ for $n \in \mathbb{Z}$.

Definition 1.1. The *Poincaré index* of Γ relative to the vector field F is defined to be this integer n.

$$I_\Gamma(F) := n = \frac{\theta(s_1) - \theta(s_0)}{2\pi}$$

If our vector field, and the parametrisation of Γ are both C^1, then we have a means of computing $I_\Gamma(F)$. We have that

$$\tan(\theta(s)) = \frac{g(x(s), y(s))}{f(x(s), y(s))},$$

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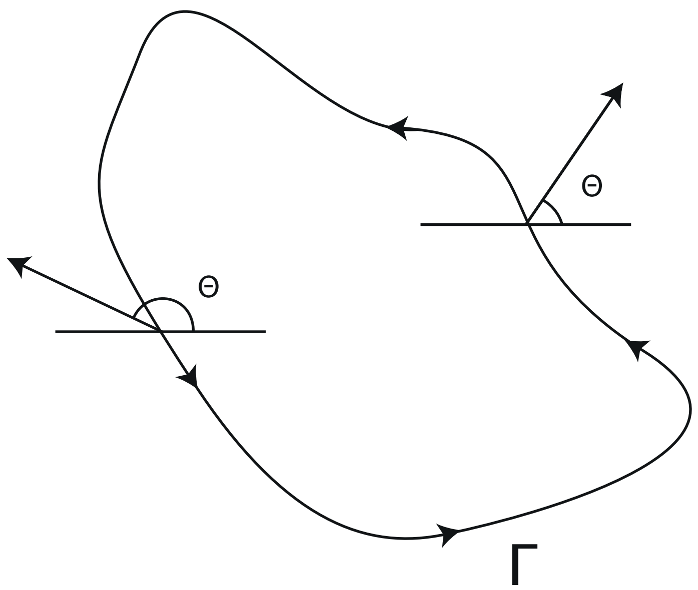

FIGURE 1

and so

$$\theta(s) = \arctan\left(\frac{g(x(s), y(s))}{f(x(s), y(s))}\right).$$

Thus we can write

$$I_\Gamma(F) = \frac{\theta(s_1) - \theta(s_0)}{2\pi} = \frac{1}{2\pi}\int_{s_0}^{s_1} \frac{\mathrm{d}}{\mathrm{d}s}\theta(s)\mathrm{d}s \equiv \frac{1}{2\pi}\oint_\Gamma \mathrm{d}\theta$$

This last term will be used for notational convenience later on. We have that

$$\frac{\mathrm{d}}{\mathrm{d}s}\theta(s) = \frac{f\frac{\mathrm{d}}{\mathrm{d}s}g - \frac{\mathrm{d}}{\mathrm{d}s}fg}{f^2 + g^2}$$

and so

$$I_\Gamma(F) = \frac{1}{2\pi}\int_{s_0}^{s_1} \frac{f\frac{\mathrm{d}}{\mathrm{d}s}g - \frac{\mathrm{d}}{\mathrm{d}s}fg}{f^2 + g^2}\mathrm{d}s = \frac{1}{2\pi}\oint_\Gamma \frac{fdg - dfg}{f^2 + g^2}$$

from the definition of the line integral. Here $df = f_x dx + f_y dy$, means the 'total exterior derivative' or sometimes just the 'total derivative' of the function f.

Example 1.1. Let's do a straightforward example to get a handle on what is going on. Let's consider the (nice, linear) vector field:

$$\dot{x} = y \qquad \dot{y} = x.$$

We let $\Gamma = (r\cos(s), r\sin(s))$ the circle of radius r around the origin, oriented counter clockwise. We have

$$I_\Gamma(F) = \frac{1}{2\pi}\oint_\Gamma \frac{y\mathrm{d}x - x\mathrm{d}y}{x^2 + y^2} = \frac{1}{2\pi}\int_0^{2\pi} \frac{-r^2\sin^2(s) - r^2\cos^2(s)}{r^2}\mathrm{d}s = -1.$$

By the way, this is independent of the size of the circle that we've chosen.

We have the following theorem:

Theorem 1.1. *Suppose that Γ is a simple closed curve. Let F be a C^2 vector field defined on Γ and its interior D_Γ. Suppose that there are no critical points of F in the set $\Gamma \cup D_\Gamma$. Then $I_\Gamma(F) = 0$.*

Proof. First, we observe that Γ being a simple closed curve means that D_Γ is what's called *simply connected.* There are many equivalent definitions of this term, which we won't really need for this class, but intuitively, you can think of simply connected regions in the plane as being 'without any holes'. Now, Green's theorem in the plane says that

$$\oint_\Gamma (P\mathrm{d}x + Q\mathrm{d}y) = \iint_{D_\Gamma} (Q_x - P_y)\,\mathrm{d}x\mathrm{d}y.$$

Okay, so we have that

$$I_\Gamma(F) = \frac{1}{2\pi}\int_{s_0}^{s_1} \frac{f\frac{\mathrm{d}}{\mathrm{d}s}g - \frac{\mathrm{d}}{\mathrm{d}s}fg}{f^2+g^2}\mathrm{d}s.$$

Now we can write

$$\frac{\mathrm{d}}{\mathrm{d}s}f = f_x\frac{\mathrm{d}x}{\mathrm{d}s} + f_y\frac{\mathrm{d}y}{\mathrm{d}s} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s}g = g_x\frac{\mathrm{d}x}{\mathrm{d}s} + g_y\frac{\mathrm{d}y}{\mathrm{d}s}.$$

Substituting this in gives

$$\begin{aligned} I_\Gamma(F) &= \frac{1}{2\pi}\oint_\Gamma \frac{fg_x - f_xg}{f^2+g^2}\mathrm{d}x + \frac{fg_y - f_yg}{f^2+g^2}\mathrm{d}y \\ &= \frac{1}{2\pi}\iint_{D_\Gamma} \frac{\partial}{\partial x}\left(\frac{fg_y - f_yg}{f^2+g^2}\right) - \frac{\partial}{\partial y}\left(\frac{fg_x - f_xg}{f^2+g^2}\right) \\ &= 0. \end{aligned}$$

Here we used Green's theorem in the plane and the fact that there are no critical points of the vector field inside D_Γ to get to the second step, and then the third equals sign is (gross but doable) algebraic computation. □

So this theorem is nice, but what it really implies is the following (perhaps more impressive) fact:

Corollary 1.1. *Let Γ_1 and Γ_2 be simple closed curves with $D_{\Gamma_1} \subseteq D_{\Gamma_2}$, and let F be defined everywhere on $\overline{D}_{\Gamma_2}$. Then if F has no critical points in the region between and including them $\overline{D_{\Gamma_2} \setminus D_{\Gamma_1}}$, then $I_{\Gamma_2}(F) = I_{\Gamma_1}(F)$.*

The point of this corollary is that if you can 'deform' one curve to another without passing through a critical point of the vector field F, then the curves must have the same indices relative to the vector field.

Proof. Let A_2A_1 be a line running from Γ_2 to Γ_1. We have by hypothesis that there are no critical points in the region bounded by the curve $\mathcal{C} := \Gamma_2 \cup A_2A_1 \cup -\Gamma_1 \cup A_1A_2$ where $-\Gamma_1$ means Γ_1 but oriented in the reverse direction, and $A_1A_2 = -A_2A_1$. Thus we have that $I_\mathcal{C}(F) = 0$. But

$$0 = I_\mathcal{C}(F) = \frac{1}{2\pi}\oint_\mathcal{C}\mathrm{d}\theta = \frac{1}{2\pi}\left(\oint_{\Gamma_2}\mathrm{d}\theta + \oint_{A_2A_1}\mathrm{d}\theta - \oint_{\Gamma_1}\mathrm{d}\theta + \oint_{A_1A_2}\mathrm{d}\theta\right)$$

$$\Rightarrow I_{\Gamma_1}(F) = I_{\Gamma_2}(F)$$

□

What this says is that *to a large extent $I_\Gamma(F)$ is independent of* Γ, or that, more or less any simple closed curve around a critical point has the same index. Thus we can drop the Γ subscript, and if the vector field is obvious from the context, we can drop the reference to that to, and just refer to I as *the index of the critical point.*

Now the game is to just go through and compute some indices, as well as some various properties.

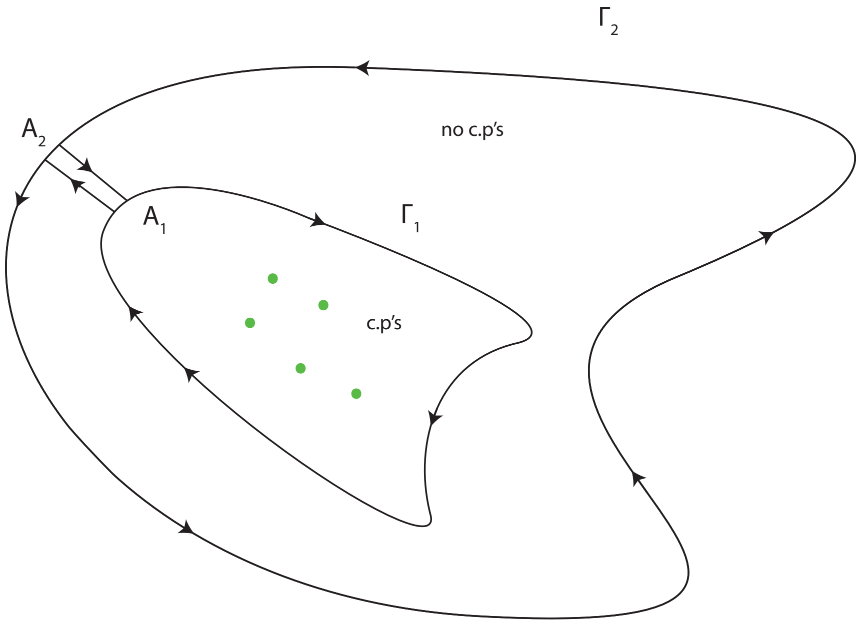

FIGURE 2

Lemma 1.1. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a* 2×2 *matrix with real coefficients, and such that* $\det(A) \neq 0$. *Then the index of the origin of the vector field* $\dot{x} = Ax$ *is*

$$I = \operatorname{sign}(\det(A)) = \frac{\det(A)}{|\det(A)|}.$$

Proof. Exercise. □

Lemma 1.2. *Suppose that* x_* *is a critical point of a planar autonomous system* $\dot{x} = F(x)$. *Suppose that the linerisation of* F *at* x_* *is hyperbolic. Then* $I_{x_*}(F) = I_0(DF(x_*))$ *that is, the index of* F *at the critical point is the index of the origin of the linearisation.*

Proof. The proof would take us too far afield, so we're not going to establish the necessary tools to prove this rigourously, but the basic idea is that the Hartman-Grobman theorem allows us to (locally) swap back and forth between a neighbourhood of a hyperbolic critical point and a neighbourhood of the origin of its linearisation without disrupting the vector field (and hence the index of the critical point) in the neighbourhood. □

These lemmata, plus the classification of 2×2 linear systems from earlier allows us to compute the indices of the linearisations of a lot of critical points. In particular, anything hyperbolic.

Lemma 1.3. *Suppose that* Γ *surrounds* n *critical points* $P_1, P_2, \ldots P_n$. *Let* I_j *be the index of point* P_j. *Then*

$$I_\Gamma = \sum_{j=1}^{n} I_j.$$

Proof. The proof is by picture. See figure 3. □

Lemma 1.4. *Let* γ *be a periodic orbit of a vector field. Then* $I_\gamma = 1$.

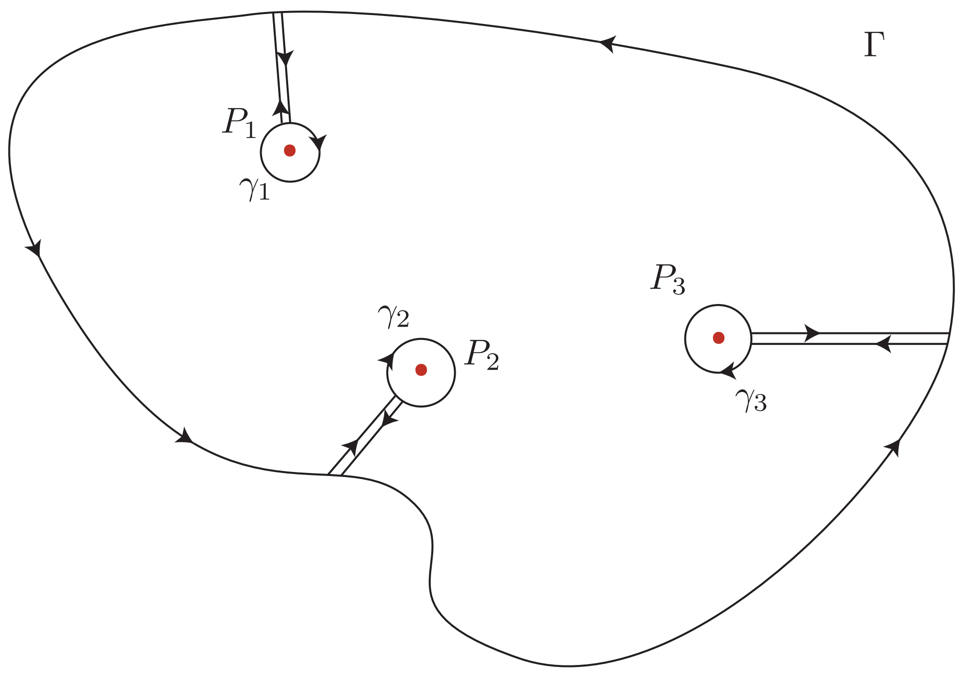

FIGURE 3. The integrals on the 'bridges' cancel out.

Proof. Again the proof is by picture. Effectively, this is because γ is a simple closed curve, and the vector field is tangent to γ by hypothesis. □

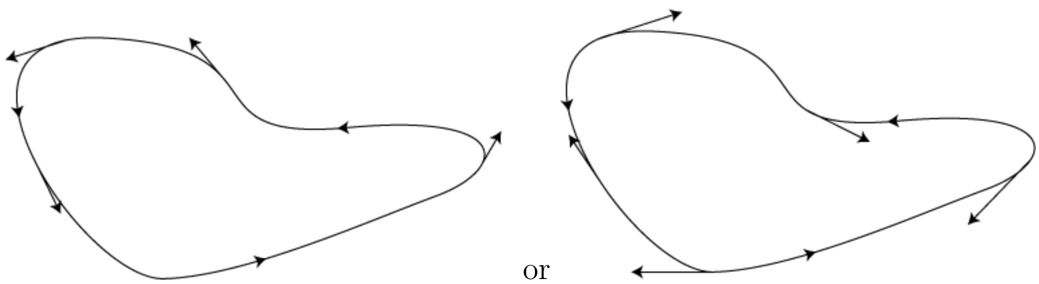

FIGURE 4. Proof of lemma 1.4. Whether the flow points in the direction of the parametrisation or against it, the vector field only rotates around once as the curve γ is traversed.

This last lemma allows you to compute the index of a nonlinear centre in the case that you can find a Hamiltonian function - that is a function whose level sets are phase curves of your vector field near the critical point.

2. The vector field at infinity

We now move onto a new topic which will give us one way of thinking about how our vector field behaves as the base points get large. We're going to define something called 'the vector field at infinity'. It should be noted right now that this is only one way to define such an object. There are others, which are not equivalent, though they are also useful and interesting objects of study.

If we denote our planar vector field by

$$\dot{x} = f(x, y) \qquad (2.1)$$
$$\dot{y} = g(x, y)$$

and we introduce new coordinates (called 'inverting the origin'):

$$\begin{aligned} x_1 &= \frac{x}{x^2 + y^2} \\ y_1 &= \frac{-y}{x^2 + y^2}, \end{aligned} \qquad (2.2)$$

and compute $\dot{x_1}$ and $\dot{y_1}$ in terms of $x, y, \dot{x}$, and $\dot{y}$ and then replace $\dot{x}$ with $f(x, y)$ and $\dot{y}$ with $g(x, y)$, and then use the fact that eq. (2.3) implies that

$$\begin{aligned} x &= \frac{x_1}{x_1^2 + y_1^2} \\ y &= \frac{-y_1}{x_1^2 + y_1^2}. \end{aligned} \qquad (2.3)$$

to get a new planar system

$$\begin{aligned} \dot{x_1} &= f_1(x_1, y_1) \\ \dot{y_1} &= g_1(x_1, y_1). \end{aligned} \qquad (2.4)$$

This new equation, eq. (2.4) is called *the vector field at* ∞ with respect to eq. (2.1). Let's do some examples.

Example 2.1. Suppose we consider the linear equation:

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x. \end{aligned} \qquad (2.5)$$

Then we have that

$$\begin{aligned} \dot{x_1} &= \frac{\dot{x}(x^2 + y^2) - x(2x\dot{x} + 2y\dot{y})}{(x^2 + y^2)^2} = \frac{-y(x^2 + y^2) - x(-2xy + 2yx)}{(x^2 + y^2)^2} = y_1 \\ \dot{y_1} &= \frac{-\dot{y}(x^2 + y^2) - +y(2x\dot{x} + 2y\dot{y})}{(x^2 + y^2)^2} = \frac{-x(x^2 + y^2) + y(-2xy + 2yx)}{(x^2 + y^2)^2} = -x_1. \end{aligned} \qquad (2.6)$$

So we have a new system in our new coordinates:

$$\begin{aligned} \dot{x_1} &= y_1 \\ \dot{y_1} &= -x_1, \end{aligned} \qquad (2.7)$$

and we see the vector field at infinity is just another linear centre, this time rotating in the other direction. See Figure 5.

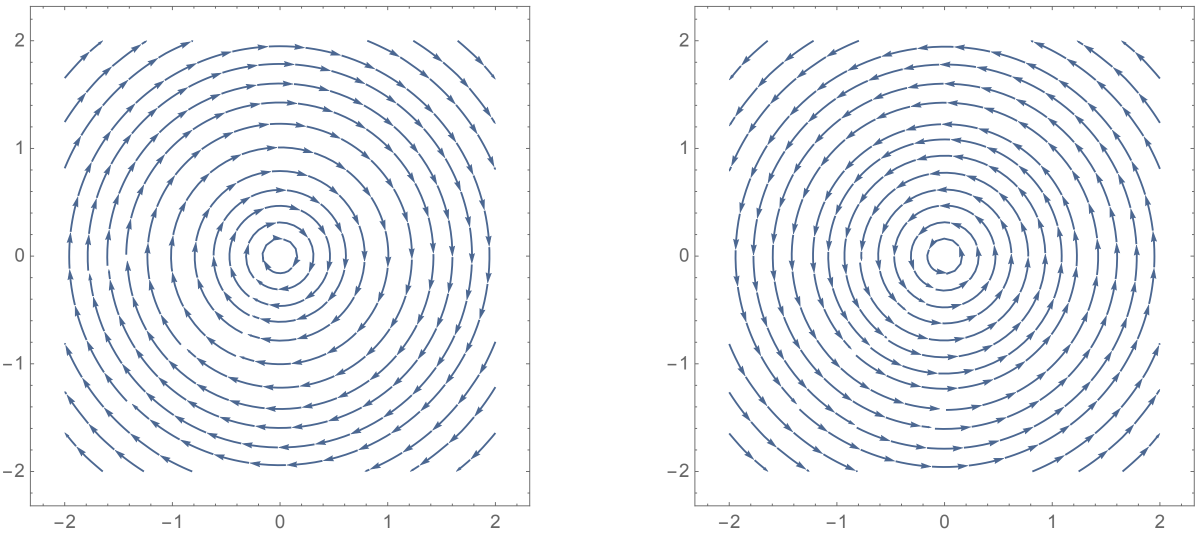

FIGURE 5. The vector field eq. (2.5) on the left, and the corresponding vector field at infinity eq. (2.6) on the right.

Example 2.2. That last example was maybe too easy. Lets consider the saddle:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x. \end{aligned} \tag{2.8}$$

Then we have this time that

$$\begin{aligned} \dot{x_1} &= \frac{3x_1^2 y_1 - y_1^3}{x_1^2 + y_1^2} \\ \dot{y_1} &= \frac{3x_1 y_1^2 - x_1^3}{x_1^2 + y_1^2}. \end{aligned} \tag{2.9}$$

Again for a picture of the vector field, see Figure 6.

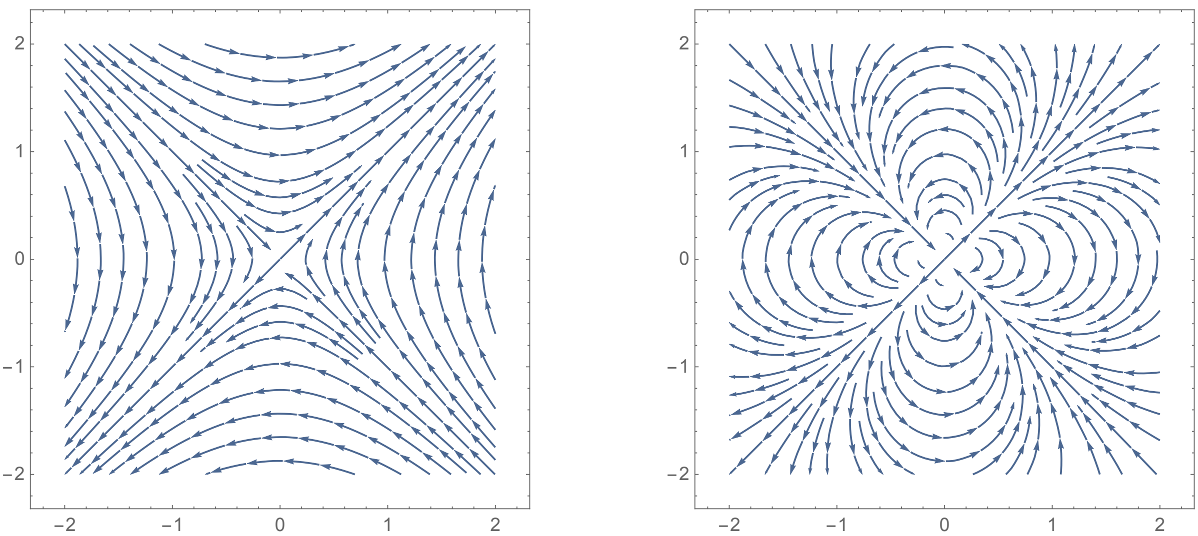

FIGURE 6. The vector field eq. (2.8) on the left, and the corresponding vector field at infinity eq. (2.9) on the right.

Example 2.3. As a final example, consider the Duffing oscillator:

$$\dot{x} = y$$
$$\dot{y} = x - x^3. \tag{2.10}$$

The vector field at infinity corresponding to eq. (2.10) is (this one is a little messier):

$$\dot{x_1} = -y_1 - \frac{2x_1^4 y_1}{(x_1^2 + y_1^2)^3} + \frac{4x_1^2 y_1}{x_!^2 + y_1^2}$$
$$\dot{y_1} = \frac{-x_1^7 + 3x_1 y_1^6 + x_1^5 + x_1^5 y_1^2 - x_1^3 y_1^2 + 5x_1^3 y_1^4}{(x_1^2 + y_1^2)^3}. \tag{2.11}$$

Again, for a plot of the vector fields, see Figure 7.

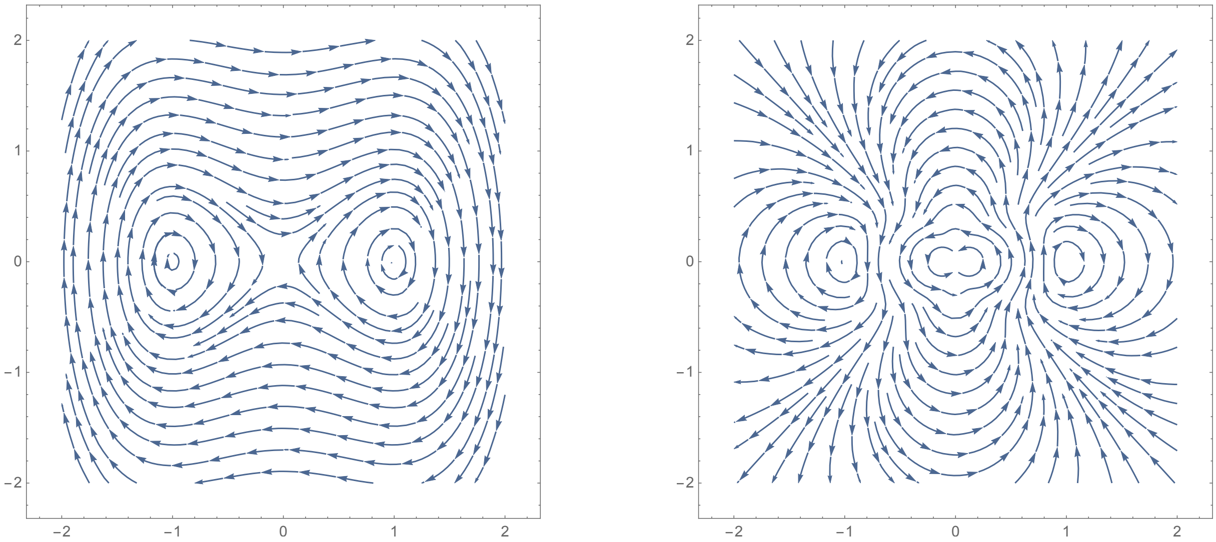

FIGURE 7. The vector field eq. (2.10) on the left, and the corresponding vector field at infinity eq. (2.11) on the right.

The inspiration for this particular transformation comes from the inversion of the (nonzero) complex plane $z_1 = \frac{1}{z}$. Sometimes it is easier to use complex notation. That is we can write $z = x + iy$ and $h(x, y) = f(x, y) + ig(x, y)$. Our ODE then becomes

$$\dot{z} = h(x, y)$$

This is not in general going to be an analytic function, but that's not a huge deal, we just need to be careful about our coordinate transformation. Now the new coordinates are

$$z_1 = \frac{1}{z}$$

which we get from setting $z_1 = x_1 + iy_1$ and then plugging in the formula for x_1 and y_1. Further we have that

$$\dot{z_1} = \frac{-1}{z^2}\dot{z} = -z_1^2 h(x, y)$$

Finally, it is worth noting that this inversion in polar coordinates is given as $r_1 = \frac{1}{r}$ and $\theta_1 = -\theta$.

Now, besides being interesting in its own right, one reason to introduce this coordinate transform is that it gives a definition of the so-called *index at infinity* of our original vector field. In many of these cases, the vector field at infinity is quite singular at the origin (in the new coordinates). However, there is still a well defined index of the vector field at 0. We'll take this to be the definition of the index of our original vector field at infinity.

Example 2.4. Suppose we have take the linear system in eq. (2.5). This is a centre, and as we have a periodic orbit around 0, we have that the index of 0 is 1. We also have that the vector field at infinity is a centre going the other way, and we have again that the index at 0 of this new vector field is 1. Thus the index at infinity of the original system is also 1.

Example 2.5. Let's consider the saddle. We have that the index of the origin is -1. Let's compute the index at infinity of this system. In this case it is easiest to switch to complex notation. We have $h = y + ix = i(x - iy) = i\bar{z}$ which is *not* an analytic function. Now we switch to our 'inverted' coordinates, we have the ODE

$$\dot{z_1} = -z_1^2 i\bar{z} = \frac{-iz_1^2}{\bar{z_1}} = -\frac{iz_1^3}{|z_1|^2}. \tag{2.12}$$

I think that the simplest way to compute the index of the origin of this guy is to switch to polar coordinates. Writing $z_1 = r_1 e^{i\theta_1}$ the expression for $\dot{z_1}$ becomes

$$\dot{z_1} = r_1 e^{i\left(3\theta_1 - \frac{\pi}{2}\right)},$$

and if we let $\gamma = e^{is}$ be the unit circle, then we can see that

$$\frac{1}{2\pi}\oint_\gamma \mathrm{d}\theta_1 = \frac{1}{2\pi}\int_0^{2\pi} 3\mathrm{d}s = 3.$$

In this instance, we interpret $\mathrm{d}\theta_1$ as the total derivative of the function which is the angle of the vector field with the horizontal, that is $3\mathrm{d}s$. Alternatively, you can compute things directly in terms of the x_1 and y_1 coordinates and compute the index of the origin. That is write

$$\begin{aligned} \dot{x_1} &= \frac{x_1^3 - 3x_1y_1^2}{x_1^2 + y_1^2} = f(x_1, y_1) \\ \dot{y_1} &= \frac{3x_1^2y_1 - y_1^3}{x_1^2 + y_1^2} = g(x_1, y_1). \end{aligned} \tag{2.13}$$

Then, letting $\gamma = (\cos(s), \sin(s))$, you compute

$$\frac{1}{2\pi}\oint_\gamma \frac{f\mathrm{d}g - \mathrm{d}fg}{f^2 + g^2} = (\text{after a fair bit of algebra}) = \frac{1}{2\pi}\int_0^{2\pi} 3\mathrm{d}s = 3.$$

Fortunately which ever way you compute it, the index of the saddle at infinity is 3.

You can (and should) verify for yourself that the indices at infinity of the Duffing oscillator is 1.

Another reason for introducing these complex coordinates is that we have the following theorem:

Theorem 2.1 (Poincaré-Hopf Index Theorem)**.** *Suppose you have a planar autonomous C^1 vector field $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = F(x, y)$ with a finite number of critical points $P_1, P_2 \ldots P_n$ with indices $I_1, I_2, \ldots, I_n$. Further denote the index at infinity by I_∞. Then we have*

$$\sum_{j=i}^{n} I_j + I_\infty = 2.$$

Proof. To compute the index at infinity, surround 0 in the z_1 plane by a curve γ_1 which contains no other equilibria besides $(0,0)$. Then because we have the relation $z_1 = \frac{1}{z}$ the curve γ_1 is transformed in the z plane into a new curve γ which encircles all the P_is. In the z_1 plane we write $\dot{z}_1 = -z_1^2 h =: h_1$. On γ_1 write $z_1 = r_1 e^{i\phi_1}$ and $h_1 = \rho_1 e^{i\theta_1}$. On γ write $h = \rho e^{i\theta}$. We have that

$$I_\infty = \frac{1}{2\pi}\oint_{\gamma_1} \mathrm{d}\theta_1.$$

We then have

$$h_1 = -z_1^2 \rho e^{i\theta} = -r_1^2 e^{i2\phi_1} e^{i\theta} = r_1^2 \rho e^{i(2\phi_1+\theta+\pi)}.$$

Thus

$$\begin{aligned} I_\infty &= \frac{1}{2\pi}\oint_{\gamma_1} \mathrm{d}\left(2\phi_1+\theta+\pi\right) = \frac{1}{2\pi}\left(\oint_{\gamma_1} 2\mathrm{d}\phi_1 + \oint_{\gamma_1} \mathrm{d}\theta\right) \\ &= \frac{1}{2\pi}\left(4\pi - \oint_{\gamma_2} \mathrm{d}\theta\right) = \frac{1}{2\pi}\left(4\pi - 2\pi\sum_{j=1}^{n} I_n\right). \end{aligned}$$

Rearranging the final equality gives the result. □

Besides being an elegant result in its own right, this theorem is very useful for computing the index at infinity - reducing it to almost a trivial task.